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LETTER TO THE EDITOR

Dilute random spin systems with finite connectivity at low temperature: solution with continuous components and longitudinal stability

J R L de Almeida†, C De Dominicis‡ and P Mottishaw§

† Departamento de Fisica, Universidade Federal de Pernambuco, 50000 Recife, Brazil
‡ Service de Physique Théorique||, CEN-Saclay, F-91191 Gif-sur-Yvette Cedex, France
§ Hewlett Packard Laboratories, Bristol, UK

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Abstract. We study dilute spin systems with finite connectivity (α) at low temperature. Near percolation ($\alpha = 1+\varepsilon$) and for small admixtures of antiferromagnetic bonds we find a solution with continuous components for the global order parameter (in contradistinction with an ansatz proposed by Mézard and Parisi and by Kanter and Sompolinsky that only contains delta functions). We relate it to an ansatz proposed by Morita and by Katsura for spins on a Bethe lattice with a $\pm J$ bond distribution. We show that such a solution is stable in the longitudinal sector.

We are interested in dilute spin-glass systems with finite connectivity, a prototype of which is the Viana–Bray (1985) model. There *all* pairs interact and the bond probability takes the form

$$\mathcal{P}(J_{ij}) = \left(1-\frac{\alpha}{N}\right)\delta(J_{ij}) + \frac{\alpha}{N} f(J_{ij}) \tag{1}$$

where α is the average connectivity of each site, N the number of sites and $f(J_{ij})$ is normalised to unity, for example,

$$f_a(J_{ij}) = a\delta(J_{ij}-J) + (1-a)\delta(J_{ij}+J) \tag{2}$$

$1-a$ being the admixture of antiferromagnetic bonds. At finite temperature the system was first studied by Viana and Bray (1985). There, only the standard order parameter $q_{\alpha_1\alpha_2}$ is needed to the lowest order in $|T-T_c|$, T_c being the critical temperature, and α_i a replica index running from 1 to n. At very low temperature the problem is more difficult for all $q_{\alpha_1\ldots\alpha_r}$ become important and one has to resort to a *global* order parameter $g\{\sigma_\alpha\}$ that has been introduced by De Dominicis and Mottishaw (1987a, b) following a step taken by Orland (1985) and Mézard and Parisi (1985) in optimisation problems. At low temperature the above system is also of interest since it maps onto the graph partitioning problem (Fu and Anderson 1986) under introduction of a constraint.

Under the assumption that there is no replica symmetry breaking $g\{\sigma_\alpha\}$ becomes a function of $\hat{\sigma} = \Sigma_\alpha\, \sigma_\alpha$ only. The equation of motion for $g(\hat{\sigma})$ is a simpler and special

|| Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique.

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case of the one derived by De Dominicis and Mottishaw (1987a, b). The $g(\hat{\sigma})$ equation of motion has also been established by Mézard and Parisi (1987) and Kanter and Sompolinsky (1987) (hereafter referred to as MP and KS respectively) who provided for it with a simple zero temperature solution. This ansatz was shown by Mottishaw and De Dominicis (1987, hereafter referred to as I) to be unstable even in the *longitudinal* sector thus suggesting that one should look for a better ansatz even within the assumption of no replica symmetry breaking.

In this letter we find such an ansatz near the percolation boundary ($\alpha \sim 1$) and we relate it to another ansatz discovered earlier, for a spin glass on a Bethe lattice with a $\pm J$ bond distribution, by Morita (1984) and Katsura (1986) (hereafter referred to as MK, see also Wong *et al* (1988)). We discuss its stability and show that as soon as the Fourier transform of $g(\hat{\sigma})$ is not localised at 0, $\pm\beta J$ as in the MP-KS ansatz, but also acquires a *continuous* component in the range $(-\beta J, +\beta J)$, then *fluctuations* around it, in the *longitudinal* sector, leave the system *stable.*

The equation of motion for the global order parameter can be written (equation (7) of I) with $\hat{\sigma} = \mathrm{i}x$,

$$G(x) = \alpha \int_{-\infty}^{+\infty} \frac{\mathrm{d}y}{2\pi} K_a(x; y) \exp G(y) \tag{3}$$

$$K_a(x; y) = \int_{-\infty}^{+\infty} \mathrm{d}J f_a(J) \int_{-\infty}^{+\infty} \mathrm{d}u \exp(-\mathrm{i}yu) \exp\{\mathrm{i}x \tanh^{-1}[\tanh u \tanh(\beta J)]\}. \tag{4}$$

Here we have used $G(x) = g(x) - \alpha$ so that $G(x)$ vanishes at the percolation transition $\alpha = 1$. We work near the transition $\alpha = 1 + \varepsilon$ and $a = 1 - \mu\varepsilon$, $G(x) \sim \mathrm{O}(\varepsilon)$. Keeping to the *lowest order* in ε we get

$$-2\varepsilon G(x) + 2\mu\varepsilon(G(x) - G(-x)) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}y}{2\pi} K_{a=1}(x; y) G^2(y). \tag{5}$$

To derive (5) we have used the identity, e.g. for $f(J)$ as in (2),

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}y}{2\pi} K_a(x; y) G(y) = aG(x) + (1-a)G(-x) \tag{6}$$

valid in the low-temperature limit.

Equation (5) looks much like the equation for a spin glass on a Bethe lattice with three nearest neighbours (see Matsubara and Sakata 1976, Katsura and Fujiki 1979, Morita 1984, Katsura 1986a, b); e.g., in Katsura (1986a, b)

$$S(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}y}{2\pi} K_{1/2}(x; y) S^2(y) \tag{7}$$

where $S(x)$ is the Fourier transform of the distribution function for the single bond effective field (note that $S(0) = 1$ whereas $G(0) = 0$). Katsura (1986a, b) deals with $a = \frac{1}{2}$, i.e. $\pm J$ bond distribution. If we are interested in the *spin-glass* sector, $G(x)$ is even and (5) becomes

$$-2\varepsilon G(x) = \int \frac{\mathrm{d}y}{2\pi} K(x; y) G^2(y) \tag{8}$$

where K can now be taken as either $K_{a=1}$, $K_{a=0}$ or $K_{a=1/2}$ for symmetry reasons. Thus we can now relate $G(x)$ and $S(x)$ via

$$S(x) = 1 + G(x)/2\varepsilon. \tag{9}$$

We now discuss the solutions to (5). Expliciting $K(x;y)$ shows (following Katsura 1986a, b) that $G(x)$ can be taken as

$$G(x)/\varepsilon = a + b\cos(\beta Jx) + d\sin(\beta Jx) + \sum_l c_l j_l(\beta Jx) + O(\varepsilon) \tag{10}$$

i.e. we expand $G(x)$ Fourier transform in Legendre polynomials. Plugging (10) into (8) yields coupled equations for a, b, d, c_l, together with the normalisation condition

$$a + b + c_0 = 0. \tag{11}$$

These equations are simplified through the use of (6). Their sum also restitutes (11). Hence, in the *spin-glass* sector, one may keep, besides (11) (cf Katsura 1986a, b)

$$-2a = a^2 + b^2/2 \tag{12}$$

$$-2c_{2l} = 2ac_{2l} + 2(1+2l)\left(2b\sum_m c_{2m}I_{2l,2m} + \sum_{m,r} c_{2m}c_{2r}I_{2l,2m,2r}\right) \tag{13}$$

$$\begin{aligned} I_{2l;2m} &= \int \frac{d\eta}{2\pi}\cos(\eta)j_{2l}(\eta)j_{2m}(\eta) \\ I_{2l,2m,2r} &= \int \frac{d\eta}{2\pi} j_{2l}(\eta)j_{2m}(\eta)j_{2r}(\eta). \end{aligned} \tag{14}$$

The couplings $I_{2l,2m}$, $I_{2l,2m,2r}$ are only accidentally null (they are listed in the useful paper by Gervois and Navelet (1987)). From (11)-(14) it is clear that one can have either

(i) $c_{2l} = 0$; $-a = b = \frac{4}{3}$ which is the MP-KS ansatz containing only three delta functions in $G(x)$ Fourier transform (at 0, $\pm\beta J$), or

(ii) all $c_{2l} \neq 0$.

The interesting remark made by MK is that keeping a few c_{2l} only (i.e. a few Legendre polynomials in the expansion of $G(x)$ Fourier transform) yields a rapidly converging approximation. We have successively:

$$\text{(iia)} \quad \text{if} \quad c_2 = c_4 = \ldots = 0 \qquad b = c_0 = 8/9,\ a = -16/9 \tag{15}$$

$$\text{(iib)} \quad \text{if} \quad c_4 = \ldots = 0 \qquad b = 0.874,\ c_0 = 0.906,\ c_2 = 0.115,\ a = -1.780. \tag{16}$$

Here (iib) is identical to the MK solution as transformed via (9). Note that the very simple (iia) only differs from it by less than 2% on b, c_0 and less than 0.1% on a.

Turning to *stability* in the *longitudinal sector* one has the eigenvalue equations

$$-(\lambda+\varepsilon)\delta G(x) - (\lambda - \mu\varepsilon)(\delta G(x) - \delta G(-x)) = \int \frac{dy}{2\pi} K(x;y)G(y)\delta G(y). \tag{17}$$

As above the most general ansatz is

$$\delta G(x) = A + B\cos(\beta Jx) + D\sin(\beta Jx) + \sum_l C_l j_l(\beta Jx) \tag{18}$$

$$0 = A + B + C_0 \tag{19}$$

where (19) expresses the orthogonality of $\delta G(x)$ to the constant eigenvector.

In the *spin-glass* phase, we note, comparing (5) and (17) the obvious, *exact* eigenvalue $\lambda = +\varepsilon$ with $\delta G \sim G$, i.e. an *even* eigenfunction. The eigenvalue equations are

$$-(\lambda+\varepsilon)A = aA + bB/2 \tag{20}$$

$$-(\lambda+\varepsilon)C_{2l}=c_{2l}A+aC_{2l}+2(1+4l)\left(\sum_m (c_{2m}B+bC_{2m})I_{2l,2m}+\sum_{m,r} c_{2m}C_{2r}I_{2l,2m,2r}\right) \tag{21}$$

with (19)-(21) replacing (11)-(13).

(i) Stability around the MP-KS ansatz. In view of the above remark it is surprising that in I we could solve *exactly* for the eigenvalues (in fact for all α and a) with what amounted to an eigenvector with (A, B, C_0) components. How could, for instance, (21) be satisfied for $l>0$, with $C_0\neq 0$, $C_2=C_4=\ldots=0$? The answer is that $I_{0,2m}=0$ for all positive m, as can easily be checked, so that (17)-(21) can be satisfied, at the same time yielding the eigenvalues $\lambda_1=\varepsilon$, $\lambda_2=-\varepsilon/3$, i.e. instability†.

(ii) Stability around the continuous (approximate) ansatz. Let us keep (A, B, C_0, C_2) as non-zero components. After eliminating A via (19) one gets (with $\Lambda\equiv 1+\lambda/\varepsilon$)

$$\begin{aligned}
&B[\Lambda-3b/2-c_0]+C_0[\Lambda-b-c_0]=0\\
&B[-c_0/2]+C_0[\Lambda-b/2-5c_0/4+c_2/16]+C_2[c_0/16+c_2/160]=0\\
&B[-23c_2/16]+C_0[5c_0/16-31c_2/32]+C_2[\Lambda-23b/16-31c_0/32+45c_2/256]=0
\end{aligned} \tag{22}$$

Here one reads the determinant (of the coefficients B, C_0, C_2) whose zeros yield the eigenvalues for cases (iia) and (iib) as follows.

(iia) In this case $c_2=c_4\ldots=0$, $b=c_0=\frac{8}{9}$ as in (15) and one gets

$$\lambda_1=1.041\ \varepsilon \qquad \lambda_2=0.309\ \varepsilon \qquad \lambda_3=1.123\ \varepsilon. \tag{23}$$

That is, as soon as continuum is introduced, the dangerous eigenvalue λ_2 becomes positive and the instability is removed. (λ_3 is, like λ_1, another approximation for the degenerate eigenvalue $+\varepsilon$).

(iib) This is the MK-like solution as in (16). Here we obtain the eigenvalues

$$\lambda_1=0.969\ \varepsilon \qquad \lambda_2=0.359\ \varepsilon \qquad \lambda_3=1.094\ \varepsilon. \tag{24}$$

Hence, by taking one extra Legendre polynomial (c_2) for the order parameter, one induces a small change (and a slow covergence towards the exact value $\lambda=\varepsilon$). Note that little is changed either if one increases the eigenvector number of components; e.g., in (iia) keeping only (A, B, C_0) one gets $\lambda_1=\varepsilon$, $\lambda_2=+\varepsilon/3$, as compared with (23).

Finally, at the spin-glass-ferro boundary, we read again on (17) the *exact* eigenvalue $\lambda=0$ (right on the boundary, i.e. $d=c_{2l+1}=0$) and $\delta G(x)$ an *odd* function. Keeping (D, C_1, C_3) as non-zero components (for simplicity we keep to (iia) i.e. $c_2=0$) we have the eigenvalue equations (with here $\Lambda=1-2\mu+\lambda/\varepsilon$):

$$\begin{aligned}
&D[\Lambda-(b+c_0)/2]+C_1[b/4+c_0/6]+C_3[b/16+319c_0/5376]=0\\
&D[3c_0/4]+C_1[\Lambda-5b/4-11c_0/16]+C_3[-3b/16+d_0/32]=0\\
&D[7c_0/16]+C_1[-7b/16+7c_0/96]+C_3[\Lambda-53b/32-291c_0/256]=0.
\end{aligned} \tag{25}$$

Focusing on the $\lambda=0$ eigenvalue that determines the boundary slope μ_c we get the following.

For a (D) component alone $\mu_c=\frac{1}{18}$. Again going from delta function components to continuum components brings a drastic change, namely,

† Ansätze containing more than three delta functions in the range ($-\beta J$, $+\beta J$) should be also treated as in I because the use of (10) would generate non-vanishing c_l for *all* l. These solutions of (8) can then be shown also to be *unstable*.

for (D, C_1) components $\mu_c = 0.171$, for (D, C_1, C_3) $\mu_c = 0.175$ close to the MP-KS value as given in I, i.e. $\mu_c = \frac{1}{6}$. Note that bringing in c_2 into (25) would only alter the answers by a few per cent.

To conclude, we now have a more reliable ansatz with (15) or (16). Stability in the transverse sector will decide whether this ansatz is the answer or whether one is to break the replica symmetry as was needed, e.g., in the Sherrington-Kirkpatrick (1975) model (with $q_{\alpha_1\alpha_2}$ only) and carried out by Parisi (1980) or as by De Dominicis and Mottishaw (1987c), (with all $q_{\alpha_1 \ldots \alpha_r}$). Note that the order parameters q_r use very little of the information contained in $G(x)$ since one has

$$q_{2r} = \varepsilon(b + 2c_0) \tag{26}$$

$$q_{2r+1} = \varepsilon\left(d - 2\sum_{l=0} c_{2l+1} C_{1/2}^{2l+1}\right). \tag{27}$$

More information is used in corrections that vanish as $\beta J \to \infty$ and, of course, in determining the fluctuation spectrum.

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References

De Dominicis C and Mottishaw P 1987a *Europhys. Lett.* **3** 87
—— 1987b *Sitges Conf. Proc., May 1986* (*Lecture Notes in Physics* **268**) ed L Garrido (Berlin: Springer)
—— 1987c *J. Phys. A: Math. Gen.* **20** L1267
Fu Y and Anderson P W 1986 *J. Phys. F: Met. Phys.* **5** 965
Gervois A and Navelet H 1988 *SIAM* to be published
Kanter I and Sompolinsky H 1987 *Phys. Rev. Lett.* **58** 164
Katsura S 1986a *Prog. Theor. Phys.* **87** 139
—— 1986b *Physica* **141A** 556
Katsura S and Fujiki S 1979 *J. Phys. C: Solid State Phys.* **12** 1087
Matsubara F and Sakata M 1976 *Prog. Theor. Phys.* **55** 672
Mézard M and Parisi G 1985 *J. Physique Lett.* **46** L771
—— 1987 *Europhys. Lett.* **3** 1067
Morita T 1984 *Physica* **125A** 321
Mottishaw P and De Dominicis C 1987 *J. Phys. A: Math. Gen.* **20** L375
Orland H 1985 *J. Physique Lett.* **46** L763
Parisi G 1980 *J. Phys. A: Math. Gen.* **13** L115
Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792
Viana L and Bray A J 1985 *J. Phys. C: Solid State Phys.* **18** 3037
Wong K Y M, Sherrington D, Mottishaw P, Dewar D and De Dominicis C 1988 *J. Phys. A: Math. Gen.* **21** L99